Assignment 2

Joe Puccio

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Collaborators: Sana Imam, Ryan Allan.

2.4

a) $(3, 4), (1, 5), (2, 5), (3, 5), (4, 5)$

b) The set array $< n, n-1, n-2, ..., 2, 1 >$. It has $\frac{(n)(n-1)}{2}$ inversions.

c) Assume that we are swapping elements with their neighbors in the insertion sort, rather than just using a temporary variable (so that each inversion in the array would necessitate one swap). There's a positive correlation between the number of inversions in the input array and the run-time of an insertion sort on the array, because the inner-loop must execute once for every inversion in the current subproblem. That is, the more inversions there are in the input array, the longer the run-time of the sort. More formally, the inner loop must execute I times where I is the number of inversions, and the outer loop must execute n times where n is the number of elements in the array, so the run time is $\Theta(I + n)$, which means that the run time increases as the number of inversions increases.

5.2-5 Given i, j indices of the array where $i < j$, let's let $C_{i,j}$ be an indicator variable for the event that (i, j) is an inversion. This means that the number of inversions in the array is $\Sigma_{i<j} C_{ij}$. Now, we know that for any given pair, the chance that the pair is inverted is $1/2$ because we're given that the array is of distinct numbers. This means that the expected value of a given $C_{ij} = 1/2$, and because we know that there are $\frac{n(n-1)}{2}$ pairs (by the sum of $1, 2...n-2, n-1$), we know that overall expected number of inversions is going to be $1/2 + 1/2 + 1/2...$ $\frac{n(n-1)}{2}$ times, so we may simply multiply the two to get that the expected value is $\frac{n(n-1)}{4}$. $\square$

5.3-3 No, the code does not produce a uniform random permutation. For each iteration of the loop, a choice of what to swap the element at i is made, where the chosen index may be from $1, ..., n$, where each element is equally probable. Because this loop cycles through n times, we know that the total number of possible ways of swapping is n^n, because as mentioned, each iteration had n choices to swap with. But, we know that the total number of *unique* permutations (that these possible ways of swapping may result in) is $n!$. We know our ways of swapping don't all result in a unique permutation, so let's write the probability that a given permutation occurs in our algorithm as k/n^n where k is an integer dependent on the permutation. So if the possibilities are to be uniformly distributed, we must have it that $k/n^n = 1/n!$, which may be rewritten as $n^n = kn!$. k must be an integer because it essentially represents the number of ways a given unique permutation can be achieved. But, in general, n^n is not evenly divisible by $n!$, so k cannot be assured to be an integer. Thus, the code does not produce a uniform random permutation.

5.3-5 Let's approach this problem intuitively. When generating the first element to be put in P, there's a

$0/n^3$ chance that our choice will be non-unique (that is, will collide with a previously chosen element). This is obvious because at that point, there are no other elements in the array. For our second element, there's a $1/n^3$ chance that our choice will be non-unique, because we could collide with our one existing element in the array. For our third, there's a $2/n^3$ chance, because we could collide with our first element or second element. We see that this pattern continues all the way up to n, where the chance that the *nth* element is non-unique is $\frac{n-1}{n^3}$. Because each of these events are independent, the chance that any of them occurs is the sum of their individual probabilities. Because their denominator is the same and their numerators are integers ranging from $1, ..., n-1$, we may write their sum as $\frac{n(n-1)}{2} \cdot \frac{1}{n^3}$ which we may simplify to $\frac{1}{2n} - \frac{1}{2n^2}$. So, this is the probability that there is a collision, but because we're considering the chance that there is *no* collision, we take the complement of this number, which is $1 - (\frac{1}{2n} - \frac{1}{2n^2}) = 1 - \frac{1}{2n} + \frac{1}{2n^2}$, and we know that this quantity is $\geq 1 - 1/n$. So we have shown that the probability that there are no collision is $\geq 1 - 1/n$. $\square$